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ANALYSIS OF COMPLEX ASSEMBLIES OF HEAT EXCHANGERS

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Abstract—The paper presents a new and general method of calculating the total effectiveness and intermediate temperatures of assemblies of heat exchangers. The assemblies may consist of associations of any types of heat exchanger.

The method utilises a transformation which relates the inlet and outlet temperatures of the fluid streams and this permits the derivation of closed form expressions. Previous expressions by Gardner, and Kays and London for assemblies of identical exchangers are shown to be special cases of the present general procedure.

The advantages of the "effectiveness-number of transfer units" approach, are stressed and it is shown that the effectiveness is the more important parameter.

NOMENCLATURE

A, heat-transfer surface (m^2);
C, heat capacity rate ($= \dot{m}c_p$) ($Js^{-1}\,°K^{-1}$);
C_{min}, C for fluid with lower $\dot{m}c_p$ ($Js^{-1}\,°K^{-1}$);
C_{max}, C for fluid with higher $\dot{m}c_p$ ($Js^{-1}\,°K^{-1}$);
${}_iC^*_{min}$, C_{min} through exchanger i ($Js^{-1}\,°K^{-1}$);
${}_iC^*_{max}$, C_{max} through exchanger i ($Js^{-1}\,°K^{-1}$);
D, static thermal transfer factor for parallel flow ($D = 1 - E(r + 1)$);
E, heat exchanger effectiveness $\left(= \dfrac{t_1 - t_0}{T_0 - t_0}\right)$;
E_t, effectiveness for the overall association of heat exchangers;
$[M]$, basic thermal static transfer matrix;
M_{11}, element of $[M]$ ($= 1 - E.r$);
M_{12}, element of $[M]$ ($= E.r$);
M_{21}, element of $[M]$ ($= E$);
M_{22}, element of $[M]$ ($= 1 - E$);
N_{tu}, number of transfer units;
P, static thermal transfer factor for counter flow ($= (1 \mp E.r)/(1 - E)$);
R, partial inversion of $[M]$ (see section 5);
T_0, inlet temperature in the association for fluid with C_{max} (°K);
T_1, outlet temperature in the association for fluid with C_{min} (°K);
U, overall unit heat transfer conductance based on unit of A ($Wm^{-2}\,°K^{-1}$);
c_p, fluid specific heat at constant pressure ($JK_g^{-1}\,°K^{-1}$);
$\dot{m}$, fluid mass flow rate ($Kg\,s^{-1}$);
n, number of heat exchangers in the association;
r, ($= C_{min}/C_{max}$);
r_i^*, ($= {}_iC^*_{min}/{}_iC^*_{max}$);
t_0, inlet temperature in the association for fluid with C_{min} (°K);
t_1, outlet temperature in the association for fluid with C_{min} (°K);
δ_1, $T_1 - t_0$ (°K);
Δ_1, $T_1 - t_1$ (°K);
$\sum_1^n$, summation of terms running from 1 to n.

Subscript

i, running index (1 to n) identifies the number on the association of a particular heat exchanger.

Superscript

*, identifies quantities pertaining to individual exchangers.

INTRODUCTION

In many practical cases a given heat-transfer duty is achieved by a combination of heat exchangers in series, parallel or series-parallel. The exchangers are often of non-identical type (or size) and a multiplicity of associations are, therefore, possible. This has prevented the derivation of a systematic treatment of calculating the overall efficiency and intermediate temperatures. The analysis of associations of identical exchangers in overall parallel or counterflow has previously been attempted by Gardner [1] and Bowman [2]; Shack [3] also considered the situation of one stream dividing equally between identical exchangers in cross-flow with the other stream. These authors obtained the overall, logarithmic-mean temperature-difference in terms of parameters which specifically related to each heat exchanger. Analyses of this type generally lead to cumbersome expressions for the logarithmic-mean temperature-difference and, if exchangers of different type are to be considered, a separate analysis is required for each assembly. A more general method is to be preferred.

An attempt to provide a more general method was reported by Kays and London [4]. In this approach, the $E - N_{tu}$ (effectiveness – number of transfer units) concept was introduced but general expressions were only obtained for associations in overall counterflow of identical exchangers. The $E - N_{tu}$ concept is, however, an important and useful one because the effectiveness, together with the heat capacity rate ratio, ($r = C_{\min}/C_{\max}$) fully describe the thermal behaviour of the exchanger. Also, if each exchanger can be characterized in terms of $E - r$, the thermal behaviour of the assembly can be expressed as a function of the same two variables; this implies that the analysis excludes any specific consideration of the type and size of the exchangers involved.

Because the heat exchanger enters the analysis only by its $E - r$, and because r follows definite rules imposed by the association considered, any two exchangers in the association are thermally equivalent provided they have the same value of E: this is true for parallel, counter, cross, mixed flow or even regenerative type exchangers.

1. OUTLINE OF THE PRESENT CONTRIBUTION

In the present contribution a general method is described for obtaining the total effectiveness and intermediate temperatures of an assembly of heat exchangers in terms of individual effectiveness and fluid heat capacity rate ratio ($r = C_{\min}/C_{\max}$). As the N_{tu} of each exchanger is not used, the analysis is independent of the particular types of exchangers involved.

Two novel concepts are introduced—the static thermal transfer matrix and the thermal transfer factor of a heat exchanger. They have been borrowed from the automatic control terminology and are described below.

If we regard a heat exchanger as a physical device or "black box", its behaviour is equivalent to an operator which transforms two input temperatures (one of each stream) to two output temperatures. The static thermal transfer matrix is that operator—when multiplied by the two input temperatures the results are the two output temperatures. This operator is a square matrix of four elements and is straightforwardly derived from the two linear algebraic equations which connect the inlet and outlet temperatures of the exchanger in terms of its effectiveness and heat capacity rate ratio.

The thermal transfer factor connects, in a similar way, the difference of input temperatures to the difference of output temperatures. It is a constant for an exchanger and is easily derived by subtracting the two rows of the transfer matrix.

By definition, the transfer factor and the transfer matrix are functions only of the effectiveness and heat capacity rate ratio of the exchanger and fully describe its thermal behaviour. Their use brings many advantages when dealing with associations of exchangers.

For the sake of illustration, let us consider an overall parallel flow association of heat exchangers. Because the two streams are common to all exchangers, r is the same for all of them. Regarding the exchangers as "black boxes", and starting from one extremity of the association, we have two input temperatures—the temperatures of the hot and of the cold stream. These two input temperatures, which we call the input vector, when multiplied by the transfer matrix, give the output temperature vector; this is the outlet temperature of the first exchanger. These outlet temperatures of the first exchanger are the inlet temperatures for the second and, consequently, the multiplication of the second exchanger by the transfer matrix gives the outlet temperatures of the second exchanger; and so on until the last exchanger. From reasoning of this type, it can be shown that the whole assembly is equivalent to one exchanger whose transfer matrix is the product of the transfer matrices of the individual exchangers in the association. Other types of association can be considered in a similar way, and the assembly is always reduced to an equivalent exchanger with its own transfer matrix and transfer factor from which the total effectiveness is found. This reduction involves a matricial product and obtaining a solution would be a tedious operation if a closed form expression could not be found. Indeed, this closed form is one of the main novelties presented: it is possible because of the particular type of matrices involved. This expression also permits a limit analysis when the number of exchangers in the association increases to any arbitrary large number which, for a particular case, proves the method of Shack [3] to give absurd results.

The main assumptions

The main assumptions employed in the analysis are summarized as follows:

(a) For each heat exchanger, the overall conductance for heat transfer—U—is a constant;

(b) fluid heat capacity rate ratio, $r = C_{min}/C_{max}$, is constant;

(c) each fluid is completely mixed at the inlet and outlet of each exchanger.

Regarding nomenclature, quantities pertaining to the fluid with higher heat capacity rate (C_{max}) are denoted by upper case letters; those pertaining to the fluid with C_{min} by lower case letters. This convention permits a more compact treatment, avoiding the duplication which would result from the explicit consideration of two expressions for the effectiveness as used, for example, by Kays and London [4].

2. FUNDAMENTAL RELATIONS†

From the definition of E and the thermal balance for the exchanger:

$$T_1 = (1 - Er)T_0 + Ert_0$$

$$t_1 = ET_0 + (1 - E)t_0$$

or, in matricial notation

$$\begin{bmatrix} T_1 \\ t_1 \end{bmatrix} = \begin{bmatrix} 1 - Er & Er \\ E & 1 - E \end{bmatrix} \begin{bmatrix} T_0 \\ t_0 \end{bmatrix} \quad (1)$$

If (T_0, t_0) are taken as the components of an input temperature vector θ_0, and (T_1, t_1) as the components of the output vector θ_1, we can write (1) as

$$\theta_1 = M\theta_0 \quad (2)$$

where M defined by (1) or (2) is the basic static thermal transfer matrix. To equation (1) or (2) corresponds the operation scheme shown in Fig. 1:

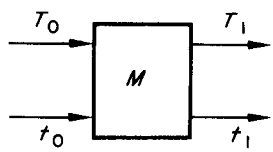

FIG. 1.

† For the readers who do not wish to go through the detailed analyses the more immediate practical results are collected in the Appendix.

by which we mean that the temperature vector on the right-hand side (output vector) is always obtained through multiplication of the vector on the left-hand side by the transfer matrix.

As we see, the basic matrix—M—completely describes the heat exchanger from a thermal point of view.

From (2) we get also

$$\theta_0 = M^{-1}\theta_1 \tag{3}$$

or

$$\begin{bmatrix} T_0 \\ \\ t_0 \end{bmatrix} = (1/D)\begin{bmatrix} 1 - E & -Er \\ \\ -E & 1 - Er \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ \\ t_0 \end{bmatrix} \tag{4}$$

where $D = 1 - E(r + 1)$ is the determinant of M.

To (3) or (4) corresponds the operational scheme of Fig. 2.

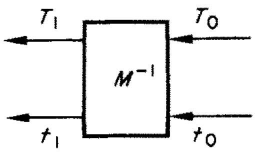

FIG. 2.

Through partial inversion of (1) there results:

$$\begin{bmatrix} T_1 \\ \\ t_0 \end{bmatrix} = \begin{bmatrix} \dfrac{1 - E(r+1)}{1 - E} & Er \\ \\ -E & 1 \end{bmatrix} \cdot \begin{bmatrix} T_0 \\ \\ t_1 \end{bmatrix} \tag{5}$$

and the operational scheme of Fig. 3.

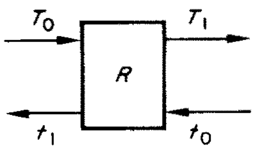

FIG. 3.

or, by total inversion of the $[R]$ matrix:

$$\begin{bmatrix} T_0 \\ \\ t_1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1 - Er} & -\dfrac{Er}{1 - Er} \\ \\ \dfrac{1}{1 - Er} & \dfrac{1 - E(r+1)}{1 - Er} \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ \\ t_0 \end{bmatrix} \tag{6}$$

to which corresponds the operational scheme of Fig. 4.

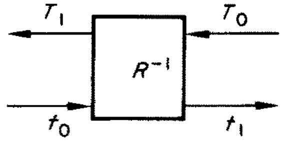

FIG. 4.

From (1) to (6) a static transfer factor for the difference between components of input and output vector is also obtained:

$$\Delta_1 = T_1 - t_1 = [1 - E(r + 1)]\Delta_0 = D(T_0 - t_0) \tag{7}$$

$$\delta_1 = T_1 - t_0 = \frac{1 - Er}{1 - E}(T_0 - t_1) = P\delta_0 \tag{8}$$

and from (7)

$$E = (1 - \Delta_1/\Delta_0)/(r + 1) = (1 - D)/(r + 1) \tag{9}$$

also from (8)

$$E = (1 - \delta_1/\delta_0)/(r - \delta_1/\delta_0) = (P - 1)/(P - r). \tag{10}$$

3. OVERALL PARALLEL FLOW

Suppose n exchanger are each characterized by its effectiveness $E_i (i = 1, 2, \ldots n)$. It follows that each one has a transfer matrix (M or R) and a transfer factor (P or D).

If they form an overall parallel flow association as shown in Fig. 5.

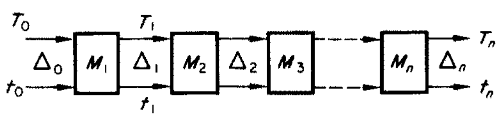

FIG. 5.

given the input vector (T_0, t_0), there results:

$$\begin{bmatrix} T_1 \\ t_1 \end{bmatrix} = [M_1] \cdot \begin{bmatrix} T_0 \\ t_0 \end{bmatrix},$$

$$\begin{bmatrix} T_2 \\ t_2 \end{bmatrix} = [M_2] . \begin{bmatrix} T_1 \\ t_1 \end{bmatrix} = [M_1] . [M_2] . \begin{bmatrix} T_0 \\ t_0 \end{bmatrix}.$$

And the exchanger n will have, as output vector

$$\begin{bmatrix} T_n \\ t_n \end{bmatrix} = [M_1] . [M_2] \ldots [M_n] . \begin{bmatrix} T_0 \\ t_0 \end{bmatrix}$$

or

$$\begin{bmatrix} T_n \\ t_n \end{bmatrix} = [N] . \begin{bmatrix} T_0 \\ t_0 \end{bmatrix}$$

where $[N] = [M_1] . [M_2] \ldots [Mn]$, $[N]$ can be found using the rules of matrix product. The usefulness of the method, however, is improved by obtaining a final expression without going to a tedious one by one product.

The assembly of n exchanger is, indeed, equivalent to only one with an equivalent effectiveness E_t. In terms of E_t we can also define a transfer matrix for the association as:

$$[N] = \begin{bmatrix} 1 - rE_t & rE_t \\ E_t & 1 - E_t \end{bmatrix} \qquad (11)$$

and a transfer factor

$$D_t = 1 - E_t(r + 1) \qquad (12)$$

or

$$\Delta_n = T_n - t_n = D_t(T_0 - t_0)\,\Delta_0$$

but

$$\Delta_1 = D_1 \Delta_0,$$
$$\Delta_2 = D_2 \Delta_1 = D_1 D_2 \Delta_0,$$
$$\Delta_n = D_1 \ldots D_n \Delta_0.$$

So

$$D_t = D_1 . D_2 \ldots D_n$$

and from (12) and (7):

$$1 - E_t(r + 1) = [1 - E_1(r + 1)] \times [1 - E_2(r + 1)] \ldots [1 - E_n(r + 1)] = \prod_{i=1}^{i=n} [1 - E_i(r + 1)]$$

or

$$E_t = \frac{1 - \prod_1^n [1 - E_i(r + 1)]}{r + 1}. \qquad (13)$$

If all exchangers have the same effectiveness, E,

$$E_t = \frac{1 - [1 - E(r + 1)]^n}{r + 1}. \qquad (14)$$

Once E_t is known, we get $[N]$ by introducing (13) in (11). From (13) or (14)

$$\lim_{n \to \infty} E_t = \frac{1}{1 + r} \qquad (15)$$

which is the maximum effectiveness of a parallel flow exchanger. So, as n increases, each heat exchanger "forgets" its own type and on the limit of $n = \infty$ the whole behaves as only a parallel flow exchanger of infinite heat transfer surface.

4. OVERALL COUNTER-FLOW

For the overall counter-flow association shown in Fig. 6.

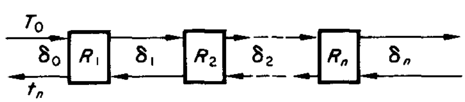

FIG. 6.

the input temperature vector is (T_0, t_n), so we have the corresponding transfer matrix R.

Using the same reasoning as before:

$$\begin{bmatrix} T_n \\ \\ t_0 \end{bmatrix} = [R_1]\ldots[R_n].\begin{bmatrix} T_0 \\ \\ t_n \end{bmatrix}$$

or

$$\begin{bmatrix} T_n \\ \\ t_0 \end{bmatrix} = [S].\begin{bmatrix} T_0 \\ \\ t_n \end{bmatrix}$$

with $[S] = [R_1]\ldots[R_n]$

$$= \begin{bmatrix} \dfrac{1 - E_t(r+1)}{1 - E_t} & \dfrac{E_t r}{1 - E_t} \\ \\ -\dfrac{E_t}{1 - E_t} & \dfrac{1}{1 - E_t} \end{bmatrix}$$

with E_t as the effectiveness of the overall assembly. Again

$$\delta_n = P_1 \ldots P_n \delta_0 = P_t \delta_0$$

and, from (8) and (10)

$$P_t = \frac{1 - E_1 r}{1 - E_1}.\frac{1 - E_2 r}{1 - E_2}\ldots = \prod_1^n \left(\frac{1 - E_i r}{1 - E_i}\right)$$

$$E_t = \frac{P_t - 1}{P_t - r} = \frac{1 - \prod\limits_1^n \left(\dfrac{1 - E_i r}{1 - E_i}\right)}{r - \prod\limits_1^n \left(\dfrac{1 - E_i r}{1 - E_i}\right)} \qquad (16)$$

from which $[S]$ is obtained in terms of individual effectivenesses.

If all the exchangers have the same effectiveness, (16) simplifies to:

$$E_t = \frac{\left(\dfrac{1 - Er}{1 - E}\right)^n - 1}{\left(\dfrac{1 - Er}{1 - E}\right)^n - r}. \qquad (17)$$

Both (16) and (17) become indeterminate when $r \to 1$. Using L'Hôpital's rule one obtains:

$$E_t = \frac{1 - \sum\limits_1^n \dfrac{E_i}{1 - E_i}}{1 + \sum\limits_i^n \dfrac{E_i}{1 - E_i}} \qquad (18)$$

and if all E_i are the same

$$E_t = \frac{nE}{1 + (n-1)E}. \qquad (19)$$

And, in any case, $\lim\limits_{n \to \infty} E_t = 1$, which shows the equivalence to a single counter-flow exchanger when $n = \infty$.

5. EXCHANGERS IN PARALLEL IN ONE OF THE STREAMS

Consider first that the stream of C_{min} is equally divided between n exchangers each with the same effectiveness; this scheme is shown in Fig. 7.

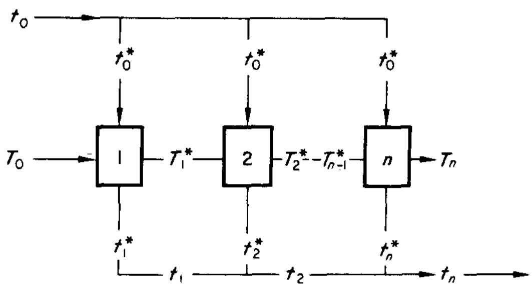

FIG. 7.

we have

$$\begin{bmatrix} T_1^* \\ \\ t_1^* \end{bmatrix} = [M].\begin{bmatrix} T_0 \\ \\ t_0 \end{bmatrix}, \begin{bmatrix} T_n \\ \\ t_n^* \end{bmatrix} = [M].\begin{bmatrix} T_{n-1}^* \\ \\ t_0^* \end{bmatrix}. \qquad (20)$$

Let us designate by M_{ij} the elements of matrix

$[M]$. There results:

$$\begin{aligned} T_1^* &= M_{11}T_0 + M_{12}t_0 \\ T_2^* &= M_{11}T_1^* + M_{12}t_0 \\ &= M_{11}^2 T_0 + M_{12}(1 + M_{11})t_0 \\ \hline T_n &= M_{11}^n T_0 + M_{12}\sum_n (1 + M_{11}^{n-1})t_0 \end{aligned}$$

but from a known result of series summation:

$$\sum_n (1 + M_{11}^{n-1}) = \frac{1 - M_{11}^n}{1 - M_{11}}$$

so

$$T_n = M_{11}^n T_0 + M_{12}\frac{1 - M_{11}^n}{1 - M_{11}} t_0 \qquad (21)$$

and, immediately, each value of t^* can be found from (20).

The overall assembly is again equivalent to one heat exchanger with the same effectiveness E_t as the whole. So, its equivalent basic transfer matrix $[F]$ of elements F_{ij} results from (1) and (21) as

$$\left.\begin{aligned} F_{11} &= 1 - E_t r = M_{11}^n \\ F_{12} &= E_t r \qquad = M_{12}\frac{1 - M_{11}^n}{1 - M_{11}} \\ F_{21} &= E_t \\ F_{22} &= 1 - E_t. \end{aligned}\right\} \quad (22)$$

From any of equations (21) and the definition of M_{11} and M_{12} results

$$E_t = \frac{1}{r}\left[1 - \left(1 - \frac{rE}{n}\right)^n\right] \qquad (23)$$

from which all the elements of F are found.

In the limit, when $r \to 0$, expression (23) becomes:

$$E_t = E \qquad (24)$$

as would be expected.

It should be noted that in relation (23) the condition $r \neq 0$, $n \to \infty$, N_{tu} finite, implies an infinite subdivision of the stream and leads to an absurd conclusion; this results from a violation of the assumptions leading to equation (1). That limit would correspond to a pure cross-flow heat exchanger where one fluid has a uniform temperature at each cross-section and a correct result could be obtained with the general approach presented by writing equation (1) in differential form and using integral calculus in a way similar to that used by Smith [5].

Except for the limiting case discussed above, expression (23) is correct for any arbitrarily large n, provided n remains finite.

It is also worth remarking that (23) corresponds to the closed form solution of an iterative method proposed by Shack [3] to deal with the cross-flow heat exchanger. Obviously the iteration does not converge to the exact solution as the present deduction shows. Besides, the usefulness of the solution found is greater than implied in Shack's analysis since no restrictions have been placed on E. So, the result is valid for any type of individual exchanger provided they have the same effectiveness and mass flow rates.

We have only considered the equal division of the stream of lower capacity rate.

If the fluid of higher heat capacity rate flows in parallel with the exchanger but such that

$$C_{\min}^* < \frac{C_{\max}}{n}, \quad \text{or} \quad r \leqslant \frac{1}{n}$$

we have the scheme of Fig. 8.

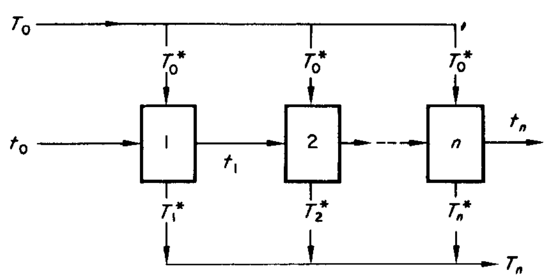

FIG. 8.

and, using the same reasoning as before, we obtain

$$t_n = M_{21}\frac{1 - M_{22}^n}{1 - M_{22}} T_0 + M_{22}^n t_0$$

and

$$E_t = 1 - (1 - E)^n. \tag{25}$$

If

$$C^*_{\min} \geqslant \frac{C_{\max}}{n}, \quad \text{or} \quad r \geqslant \frac{1}{n}$$

we have again (21) for the intermediate temperatures but for the equivalent exchanger T and t are reversed.

For the equivalent exchanger results

$$E_t = 1 - \left(1 - \frac{E}{nr}\right)^n \tag{26}$$

where, as before, r refers to the undivised streams and E to the effectiveness of the individual exchanger.

If the exchangers do not have the same effectiveness but have the same M_{11}, which means the same $E_i r_i^*$ where r_i^* refers to the heat capacity rate ratio of the i exchanger, we would have for the subdivision of the lower capacity rate stream between the exchangers

$$E_t = \frac{1 - (1 - r_i^* E_i)^n}{r} \tag{27}$$

and if $r = 0$

$$E_t = \sum_1^n \frac{{}_iC^*_{\min}}{C_{\min}} E_i \tag{28}$$

where ${}_iC^*_{\min}/C_{\min}$ is the fraction of the stream which comes through the exchanger i.

If the subdivision is on the stream of higher heat capacity rate but such that

$$\text{lower } {}_iC^*_{\max} > C_{\min} \tag{29}$$

then

$$E_t = 1 - (1 - E_i r_i^*). \tag{30}$$

But when

$$\text{higher } {}_iC^*_{\min} < C_{\min} \tag{31}$$

the total effectiveness becomes independent of the stream subdivision provided all the exchangers have the same effectiveness, i.e.

$$E_t = 1 - (1 - E)^n. \tag{32}$$

All the conclusions in this section are easily derived from the previous analysis and can, in a similar way, be extended when (29) or (32) are not verified.

For the more general case of exchangers with different r and E with one or both streams subdivided between them the same general method applies. As should be expected, however, the final expressions become more involved and have not been evaluated in this paper.

6. MIXED FLOW

Mixed flow assemblies are those where some sections are in overall counter flow and others in overall parallel flow, irrespective of the type of exchangers involved.

Let us exemplify such assemblies which, taking as reference the stream of $C_{\max}$, would assume the functional scheme shown by Fig. 9.

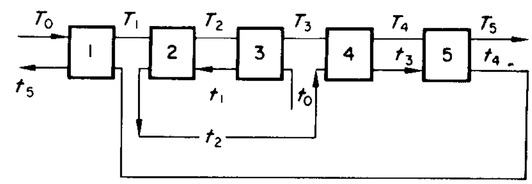

FIG. 9.

The exchangers 2 and 3 are in overall counter flow, so, we can construct for them the equivalent matrix, $[R]$. For the exchangers 4 and 5, in overall parallel flow we can also construct its equivalent matrix $[M]$. In this way we reduce the assembly to the equivalent one shown in Fig. 10.

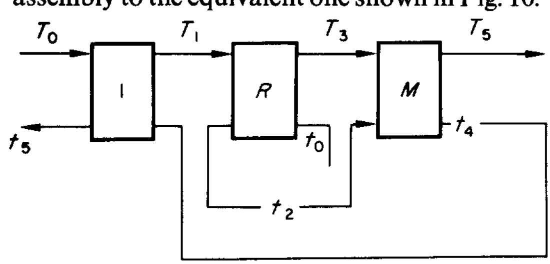

FIG. 10.

For the assembly R, we invert partially its matrix to the basic matrix $[M]$ which we should call $[M_R]$ and obtain the scheme shown in Fig. 11.

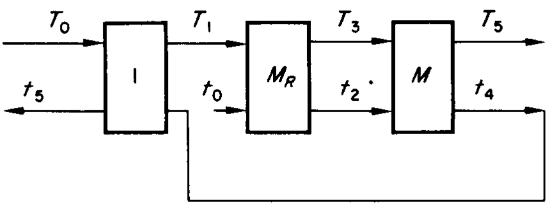

FIG. 11.

$[M_R]$ and $[M]$ correspond again to overall parallel flow, so we can obtain at once its product $[P] = [M_R] . [M]$; this transforms the assembly in that of Fig. 12.

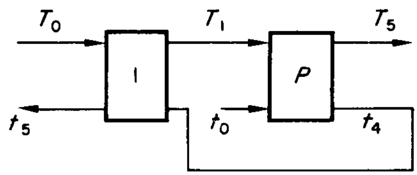

FIG. 12.

$[P]$ has the structure of the basic transfer matrix. Inverting it partially to the structure of an $[R]$ matrix, which we call $[P_R]$, results the scheme shown in Fig. 13.

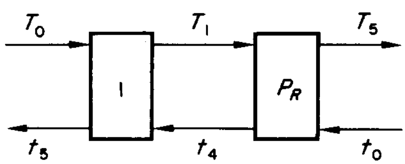

FIG. 13.

Forming now the $[R]$ matrix of (1) the assembly is reduced by the overall counter-flow association to the equivalent exchanger shown in Fig. 14.

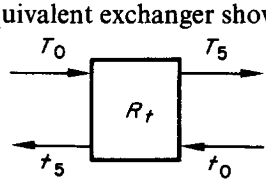

FIG. 14.

It should be remarked that all the expressions for obtaining the product of matrices or its partial inversion have already been presented. Indeed, at each stage, only the equivalent effectiveness is needed since the assumption that r is constant implies that all the matrices can be deduced directly from it. The final expressions can be complex, particularly if the values of E are different for each exchanger but, in any case, they are found in a very straightforward way.

After reducing the association to an equivalent exchanger the unknown input and output temperatures are found at once. Inverting successively the operations performed, we obtain all the intermediate temperatures.

7. CONCLUSIONS

The concept of transfer factor and transfer matrix has led to a direct method for finding the equivalent effectiveness of an association. In this way, the expressions previously given by Kays and London have been generalized for exchangers of different effectiveness. It is also shown that the important parameter in the analysis is E and not the N_{tu}.

The iterative expression of Shack has been given a closed form, and generalized for any type of exchanger.

The indirect transfer type exchangers, previously analysed by Kays and London [4] can easily be dealt with by the present method.

Besides its interest as a general method, some practical results of immediate application for designers have been derived and are presented in the Appendix. These include the expressions for the effectiveness of an association in overall counter-flow, overall parallel flow, a combination of parallel-counter flow (mixed flow) and for the subdivision of one of the streams between the heat exchangers.

From a practical point of view, if the association is built from existing exchangers and the designer requires the outlet temperatures from a knowledge of the inlet values, the method has all the known advantages of the $E - N_{tu}$ approach for a single exchanger.

If the task is to achieve a fixed total effectiveness, the method greatly simplifies the analysis because it is independent of the type or size of heat exchangers involved; this frees the designer from the requirement of equal exchangers.

Finally, it should be stressed, the method reduces to a matrix product and the results are easily obtained because the $E - N_{tu}$ concept has only led to linear algebraic equations instead of the non-linear ones characteristics of the logarithmic-mean temperature difference approach.

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APPENDIX

Immediate Practical Results

For the readers who do not wish to go through the detailed analysis, the more immediate practical results are collected in this section.

A.1 *Overall parallel flow association* (*Fig.* 5)

For the total or equivalent effectiveness of the whole assembly we obtain

$$E_t = \frac{1 - [1 - E_t(r+1)]\,[1 - E_2(r+1)]\ldots[1 - E_i(r+1)]}{r+1}$$

or, in more compact form

$$E_t = \frac{1 - \prod_1^n [1 - E_i(r+1)]}{r+1}$$

where $E_i (i = 1, 2, \ldots n)$ stands for the effectiveness of each exchanger in the association, which are numbered from 1 to n.

If the exchangers have the same effectiveness the above expressions simplify to:

$$E_t = \frac{[1 - E(r+1)]^n}{r+1}.$$

A.2 *Overall counter flow association* (*Fig.* 6):

If $r \neq 1$, the total effectiveness is

$$E_t = \frac{1 - \prod_1^n \left(\frac{1 - E_i r}{1 - E_i}\right)}{r - \prod_1^n \left(\frac{1 - E_i r}{1 - E_i}\right)}.$$

If $r = 1$

$$E_t = \frac{1 - \sum_1^n \frac{E_i}{1 - E_i}}{1 + \sum_1^n \frac{E_i}{1 - E_i}}.$$

When the exchangers have all the same effectiveness the above expressions simplify to the known results of Kays and London [4]:

$$E_t = \frac{\left(\frac{1 - Er}{1 - E}\right)^n - 1}{\left(\frac{1 - Er}{1 - E}\right)^n - r} \quad (r \neq 1)$$

$$E_t = \frac{nE}{1 + (n-1)E} \quad (r = 1).$$

A.3 *One of the streams subdivided between exchangers* (*Fig.* 3)

First assume that the exchangers have the same effectiveness, with one of the streams equally divided between them.

If the subdivision is on the stream with lower heat capacity rate, the total effectiveness for the association is

$$E_t = \frac{1}{r}\left[1 - \left(1 - \frac{E\,.\,r}{n}\right)^n\right] \quad (r \neq 0)$$

$$E_t = E. \quad (r = 0)$$

If the subdivision is on the stream of higher heat capacity, but such that

$$r \leqslant \frac{1}{n}$$

then

$$E_t = 1 - (1 - E)^n$$

when

$$r \geqslant \frac{1}{n}$$

$$E_t = 1 - \left(1 - \frac{E}{nr}\right)^n.$$

If the exchangers do not have the same effectiveness neither one stream equally divided between them but $E_i r_i^*$ is constant for each one, the following results can be deduced for the subdivision on the stream of lower capacity rate:

$$E_t = \frac{1}{r}[1 - (1 - E_i\,.\,r_i^*)] \quad (r \neq 0)$$

$$E_t = \frac{1}{C_{\min}} \sum_1^n {}_iC^*_{\min}\,.\,E_i.$$

If the subdivision is on the stream with higher heat capacity rate and:

$$\text{lower } {}_iC^*_{\max} \geqslant C_{\min}$$

then

$$E_t = 1 - (1 - E_i r_i^*)$$

but when

$$(\text{higher } {}_iC^*_{\min}) \leqslant C_{\min}$$

the effectiveness being equal for each exchanger, the total effectiveness becomes independent of the stream division and is given by

$$E_t = 1 - (1 - E)^n.$$

A.4 *Mixed flow*

Considering mixed flow to be a combination of parallel and counter flow, as for example the arrangement shown in Fig. 15

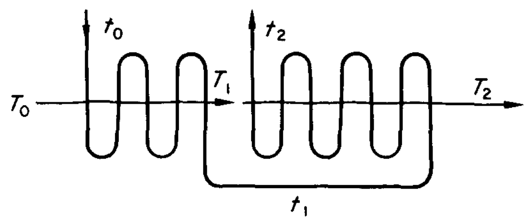

FIG. 15.

or, in symbolic representation, shown in Fig. 16

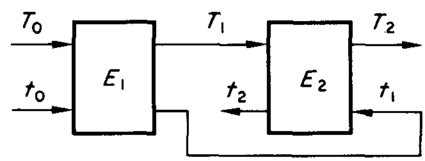

FIG. 16.

the total effectiveness is

$$E_t = \frac{1 - [1 - E_1(r + 1)]\,[1 - E_2(r + 1)]}{r + 1};$$

this is the same as for overall parallel flow. We remark, again, that this is obtained very easily because the true flow arrangement has already entered through the effectiveness of the exchangers.

With the "black box" concept the same arrangement can be expressed as shown in Fig. 17.

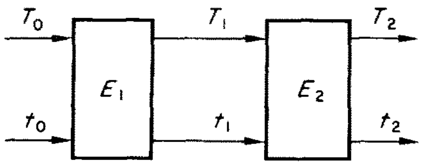

FIG. 17.

clearly showing that the unknown temperatures on the right side of the "box" are calculated in chain from those on the left.

Having been able to find the equivalent exchanger for an overall parallel or counter flow arrangement a complex association can always be reduced to the above example of mixed flow.

This is easier for the total effectiveness. For the intermediate temperatures partial association can be used and this reduces that part to an equivalent exchanger for which the unknown temperatures are easily found. This can readily be understood from the matricial analysis presented in the text.

Résumé—L'article présente une méthode nouvelle et générale de calcul de l'efficacité totale et des températures intermédiaires de groupements d'échangeurs de chaleur. Les groupements peuvent consister en associations de n'importe quel type d'échangeur de chaleur.

La méthode utilise une transformation qui relie les températures d'entrée et de sortie des écoulements de fluides et permet d'obtenir des expressions analytiques. Les expressions antérieures de Gardner et de Kays et London pour des groupements d'échangeurs identiques sont des cas spéciaux des procédés généraux actuels.

Les avantages de la méthode "efficacité—nombre d'unités de transfert" sont mis en valeur et l'on montre que l'efficacité est le paramètre le plus important.

Zusammenfassung—Die Untersuchung liefert eine neue und allgemeine Methode zur Berechnung des Gesamtwirkungsgrades und der Zwischentemperaturen einer Zusammenschaltung von Wärmeaustauschern Diese Systeme Können dakei aus Warmeaustauchern der verschiedensten Art aufgebaut sein.

Die Methode stützt sich auf eine Transformation welche die Einund Austrittstemperaturen der Fluidströme verknüpft und auf diese Weise die Ableitung geschlossener Ausdrücke ermöglicht. Ferner wird gezeigt, dass ein bereits früher von Gardner und Kays und London für ein System identischer Austauscher angegebener Formalismus Spezialfälle des hier vorgelegten allgemeinen Verfahrens beschreibt.

Аннотация—В статье дается новый и обобщенный метод расчета суммарной эффективности и промежуточных температур в узлах теплообменников. Узлы могут состоять из блоков любых типов теплообменника.

Метод основан на преобразовании, которое связывает температуры потоков жидкости на входе и на выходе, что позволяет вывести выражения в замкнутом виде. Показано, что ранее полученные Гарднером и Кейсом и Лондоном выражения для узлов идентичных теплообменников являются частными случаями представленных в настоящей статье общих методов.

Подчеркиваются преимущества подхода «эффективность и число единиц переноса». Показано, что эффективность является наиболее важным параметром.